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On the partition function of the six-vertex model with domain wall boundary conditions

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Abstract
The six-vertex model on an $N \times N$ square lattice with domain wall boundary conditions is considered. A Fredholm determinant representation for the partition function of the model is given. The kernel of the corresponding integral operator is of the so-called integrable type, and involves classical orthogonal polynomials. From this representation, a 'reconstruction' formula is proposed, which expresses the partition function as the trace of a suitably chosen quantum operator, in the spirit of corner transfer matrix and vertex operator approaches to integrable spin models.

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1. Introduction

The six-vertex model on a square lattice with domain wall boundary conditions (DWBC) was introduced in [1] and subsequently solved in [2], where a determinant formula for the partition function was obtained and proven (see also [3]). This model, in its inhomogeneous formulation (i.e. with the vertex weights given as suitable functions of the position of the vertex on the lattice), naturally arises when investigating correlation functions of quantum integrable models in the framework of the quantum inverse scattering method. In its homogeneous version, the model admits the usual interpretation as a model of statistical mechanics with fixed boundary conditions, and may be seen as a variation of the original six-vertex model with periodic boundary conditions [4–7], which has been for decades a paradigmatic one in statistical mechanics [8].

The DWBC version of the six-vertex model has however several nontrivial peculiarities which render it worthy of further investigation. Firstly, the model with these very specific fixed boundary conditions enjoys interesting connections with some important issues of enumerative combinatorics, such as alternating sign matrices [9–11] and domino tilings [12, 13]. Moreover, it appears that, even in the thermodynamic limit, bulk quantities, such as the bulk free energy,

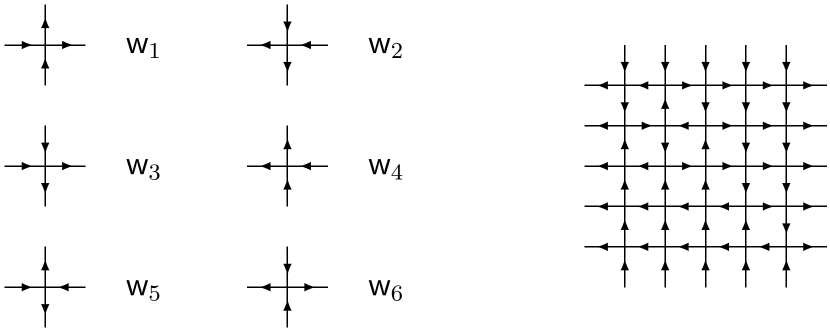

Figure 1. The six allowed types of vertices, their weights and one of the possible configurations in the model with the domain wall boundary conditions for $N = 5$.

are indeed sensitive to the choice of boundary conditions [14, 15] (see also [16]). Finally, the determinant representation given in [2, 3] for the partition function, and analogous ones recently presented for the boundary one-point correlation functions (polarizations) [17], are rather implicit and turn out to be too intricate for any further, more explicit, answer, except in very particular cases. Alternative equivalent representations for the partition function and polarizations would therefore be highly desirable to address several problems, such as further alternating sign matrices weighted enumerations, or extensions of the Arctic Circle theorem [13] beyond the free-fermion point.

The model is formulated on a square lattice with arrows on edges. The only admitted configurations are such that there are always two arrows pointing away from, and two arrows pointing into, each lattice vertex; each vertex can therefore be in one out of six different possible states, a Boltzmann weight w_i being assigned to each vertex, according to its state i ($i = 1, \ldots, 6$). We shall consider here the homogeneous version of the model, where the Boltzmann weights are site independent. The DWBC are imposed on the $N \times N$ square lattice by fixing the direction of all arrows on the boundaries as follows: the vertical arrows on the top and bottom of the lattice point inwards, while the horizontal arrows on the left and right sides point outwards. The correspondence between the Boltzmann vertex weights and the arrow configurations, and a typical configuration of the model with DWBC, are shown in figure 1.

The partition function is obtained by summing over all possible arrow configurations, compatible with the imposed DWBC, each configuration being assigned its Boltzmann weight, given simply as the product of all the corresponding vertex weights:

$$Z_N = \sum_{\text{DWBC configurations}} \prod_{i=1}^{6} \mathsf{w}_i^{n_i}. \tag{1.1}$$

Here n_i denotes the number of vertices in state i, i.e. with Boltzmann weight w_i, in each configuration, and $\sum_{i=1}^{6} n_i = N^2$. The six-vertex model with DWBC is usually considered with its weights invariant under inversion of all arrows, and thus with only three distinct weight functions, denoted as a, b and c,

$$\mathsf{w}_1 = \mathsf{w}_2 \equiv \mathsf{a} \qquad \mathsf{w}_3 = \mathsf{w}_4 \equiv \mathsf{b} \qquad \mathsf{w}_5 = \mathsf{w}_6 \equiv \mathsf{c}. \tag{1.2}$$

We shall use the following parametrization for the weight functions

$$\mathsf{a} = \sin(\lambda + \eta) \qquad \mathsf{b} = \sin(\lambda - \eta) \qquad \mathsf{c} = \sin 2\eta. \tag{1.3}$$

In terms of this parametrization the result of [3] for the partition function reads

$$Z_N = \frac{[\sin(\lambda - \eta)\sin(\lambda + \eta)]^{N^2}}{\prod_{k=1}^{N-1}(k!)^2}\, \det_N H \tag{1.4}$$

where H is an $N \times N$ Hänkel matrix, with entries

$$H_{jk} = \frac{\partial^{j+k}}{\partial\lambda^{j+k}} \frac{\sin 2\eta}{\sin(\lambda - \eta)\sin(\lambda + \eta)}. \tag{1.5}$$

Here and in the following we use the convention that indices of $N \times N$ matrices run over the values $j, k = 0, 1, \dots, N-1$. Formula (1.4) for the partition function, which was originally obtained within the quantum inverse scattering method, will be referred to as the Hänkel determinant representation.

The purpose of the present paper is to give some other equivalent representations for the partition function. The emphasis will be made on the representation in terms of the Fredholm determinant of some linear integral operator of integrable type, in the sense of paper [18]. Representations of such type have been proven to be powerful tools in many areas of mathematical physics, ranging from the theory of random matrices to the asymptotics of orthogonal polynomials. Among them is the theory of correlation functions of quantum integrable models [19], the area of origin of the six-vertex model with DWBC itself.

The general procedure which we shall follow to build Fredholm determinant formula for the partition function has been suggested in [20]; in contrast to that paper, we shall however apply this procedure directly to the Hänkel determinant representation (1.4). In this way, the Hänkel structure is preserved, and the integrability of the integral operator in the Fredholm determinant is ensured by construction. We therefore propose a simple factorization for the determinant of the Hänkel matrix H appearing in (1.4). This factorization, quite natural in the construction of a Fredholm determinant representation for the partition function, allows moreover to identify a core term in the factorized form of (1.4), with all other factors being trivial, and disappearing with a mere redefinition of vertex weights. The core term turns out to correspond exactly to the partition function for the six-vertex model when its R-matrix is specialized to the $U_q(\mathfrak{sl}_2)$-invariant R-matrix [21], i.e. when its vertex weights are chosen accordingly with its underlying quantum group symmetry. This core term, which we shall denote as $\tilde{Z}_N$, gives rise to the Fredholm determinant, thanks to standard techniques from the theory of orthogonal polynomials.

Together with this Fredholm determinant representation, we readily get for $\tilde{Z}_N$ an equivalent representation as the ordinary determinant of an $N \times N$ symmetric matrix, whose entries can be explicitly evaluated. From this last representation we propose a 'reconstruction' formula which expresses the partition function as a trace of some quantum operator, which, in the large N limit, turns into the exponential of the boost operator for free fermions on a lattice. This recalls analogous formulae which appear in the framework of corner transfer matrix and vertex operator approaches to integrable lattice models [8, 22]. For finite N, a corresponding 'reconstruction' formula for $\tilde{Z}_N$ can also be written, as the trace (over the Fock space of N canonical fermions) of a product of exponentials of local operators.

2. The factorization of the Hänkel determinant

In our way to build a Fredholm representation for the partition function Z_N we shall follow the procedure suggested in [20], but applying it directly to the Hänkel determinant representation (1.4). As a consequence, the Hänkel structure is preserved, and the integrability of the integral operator in the Fredholm determinant is ensured by construction. In this section, we shall discuss a specific factorization (a somewhat trivial one, but nevertheless basic for what follows) for the determinant of the Hänkel matrix H appearing in (1.4).

All the different equivalent representations for the partition function derived in this paper stem essentially from the following simple observation: using the identity

$$\frac{\sin 2\eta}{\sin(\lambda-\eta)\sin(\lambda+\eta)} = \cot(\lambda-\eta) - \cot(\lambda+\eta) \tag{2.1}$$

the matrix H can be naturally written as a difference of two matrices

$$H = A_- - A_+ \qquad A_\pm = A|_{\phi=\phi_\pm} \qquad \phi_\pm = \lambda \pm \eta \tag{2.2}$$

where the matrix A can be chosen to be

$$A_{jk} = \frac{\partial^{j+k}}{\partial\phi^{j+k}}[\cot\phi - \mathrm{i}]. \tag{2.3}$$

Our choice of the additive constant $-\mathrm{i}$ will be explained below; its role is to ensure invertibility of matrix (2.3) for any complex value of ϕ. The structure of (2.2) suggests to factorize the determinant of H, for instance, as follows

$$\det_N H = \det_N(A_-)\det_N(I - A_-^{-1}A_+). \tag{2.4}$$

The most evident consequence of such factorization is of course the relatively natural and straightforward emergence of a Fredholm determinant representation for the partition function, the Fredholm determinant being related to the second factor in (2.4). The proposed factorization however suggests more: as we shall discuss in detail in the following (see section 4), a 'reconstruction' formula can be deduced, which allows us to represent the partition function as the trace of a suitably chosen quantum operator, in the spirit of corner transfer matrix and vertex operator approaches to integrable spin models.

The analysis of factorization (2.4) relies essentially on the properties of matrix A and in particular of its determinant. To this purpose standard techniques relating Hänkel matrices to orthogonal polynomials [23] will be exploited. Let us assume that entries of some generic $N \times N$ Hänkel matrix A are given as

$$A_{jk} := \int_{-\infty}^{\infty} x^{j+k}\omega(x)\,\mathrm{d}x. \tag{2.5}$$

Let us moreover suppose that there exist a (complete) set of polynomials $p_n(x)$, orthonormal with respect to the measure $\omega(x)\,\mathrm{d}x$:

$$\int_{-\infty}^{\infty} p_j(x)p_k(x)\omega(x)\,\mathrm{d}x = \delta_{jk}. \tag{2.6}$$

Then, calling κ_n the leading coefficient of $p_n(x)$,

$$p_n(x) = \kappa_n x^n + \cdots \qquad \kappa_n \neq 0 \tag{2.7}$$

the determinant of matrix (2.5) is simply given as

$$\det_N A = \prod_{j=0}^{N-1}\kappa_j^{-2}. \tag{2.8}$$

Of course, this formula turns out to be useful provided that the set of orthogonal polynomials associated with the measure $\omega(x)\,\mathrm{d}x$ can be identified. In the case of matrix H given by (1.5) appropriate polynomials are not available and the previous scheme cannot be fulfilled for generic values of vertex weights. However, the matrix defined in equation (2.3) is much simpler and the scheme can be fulfilled explicitly.

The entries of matrix (2.3) being periodic in $\mathrm{Re}\,\phi$, we may restrict ourselves to consider values of ϕ varying over the vertical strip $0 \leqslant \mathrm{Re}\,\phi < \pi$ (with the point $\phi = 0$ excluded). In this region we may use

$$\cot\phi = \text{v.p.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\phi x}}{1 - \mathrm{e}^{\pi x}}\, \mathrm{d}x \tag{2.9}$$

to write the entries of matrix A, equation (2.3), in the form (2.5) with

$$\omega(x) = \frac{\mathrm{e}^{\phi x}}{1 - \mathrm{e}^{\pi x} + \mathrm{i}\,0}. \tag{2.10}$$

The polynomials $p_n(x)$ (depending on ϕ as a parameter), associated with the matrix A, should therefore satisfy

$$\int_{-\infty}^{\infty} p_j(x) p_k(x) \frac{\mathrm{e}^{\phi x}}{1 - \mathrm{e}^{\pi x} + \mathrm{i}\,0}\, \mathrm{d}x = \delta_{jk}. \tag{2.11}$$

To identify the explicit form of these polynomials we shall now reexpress the orthogonality condition in such a way that the integration contour, though still being the real axis, has no singularity in its vicinity. This can be achieved by shifting the integration contour $\mathbb{R} \to \mathbb{R} - \mathrm{i}$ and simultaneously relabelling the integration variable: $x \to x - \mathrm{i}$. The orthogonality condition now reads

$$\mathrm{e}^{-\mathrm{i}\phi} \int_{-\infty}^{\infty} p_j(x - \mathrm{i}) p_k(x - \mathrm{i}) \frac{\mathrm{e}^{\phi x}}{1 + \mathrm{e}^{\pi x}}\, \mathrm{d}x = \delta_{jk}. \tag{2.12}$$

Rewriting the weight function as

$$\frac{\mathrm{e}^{\phi x}}{1 + \mathrm{e}^{\pi x}} = \frac{1}{2\pi} \Gamma\left(\frac{1 - \mathrm{i}x}{2}\right) \Gamma\left(\frac{1 + \mathrm{i}x}{2}\right) \mathrm{e}^{(\phi - \pi/2)x} \tag{2.13}$$

and comparing (2.12) with the orthogonality condition of Meixner–Pollaczek polynomials [24]

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} P_n^{(\lambda)}(x; \phi)\, P_m^{(\lambda)}(x; \phi)\, \Gamma(\lambda - \mathrm{i}x) \Gamma(\lambda + \mathrm{i}x)\, \mathrm{e}^{(2\phi - \pi)x}\, \mathrm{d}x = \frac{\Gamma(n + 2\lambda)}{n!(2\sin\phi)^{2\lambda}} \delta_{nm} \tag{2.14}$$

where

$$P_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!}\, \mathrm{e}^{\mathrm{i}n\phi}\, {}_2F_1\left(\left. \begin{matrix} -n, \lambda + \mathrm{i}x \\ 2\lambda \end{matrix} \right| 1 - \mathrm{e}^{-2\mathrm{i}\phi} \right) \tag{2.15}$$

we readily identify the polynomials in question with nothing but the Meixner–Pollaczek polynomials, where the parameter λ (not to be confused with the variable λ entering the parametrization of the vertex weights) must be specialized to the value $\lambda = 1/2$.

Therefore, the polynomials $p_n(x)$ satisfying (2.11) are

$$p_n(x) = \mathrm{e}^{\mathrm{i}\phi/2} \sqrt{\sin\phi}\, P_n^{(1/2)}\left(\frac{x + \mathrm{i}}{2}; \phi \right) \tag{2.16}$$

or, due to (2.15), more explicitly

$$p_n(x) = \mathrm{e}^{\mathrm{i}(n+1/2)\phi} \sqrt{\sin\phi}\, {}_2F_1\left(\left. \begin{matrix} -n, \mathrm{i}x/2 \\ 1 \end{matrix} \right| 1 - \mathrm{e}^{-2\mathrm{i}\phi} \right). \tag{2.17}$$

From this expression the highest coefficient κ_n of the polynomial $p_n(x)$ can be explicitly evaluated as

$$\kappa_n(\phi) = \frac{\mathrm{e}^{\mathrm{i}\phi/2} (\sin\phi)^{n+1/2}}{n!} \tag{2.18}$$

and, due to formula (2.8), we readily get for the determinant of the matrix A:

$$\det_N A = \frac{\mathrm{e}^{-\mathrm{i}N\phi}}{(\sin\phi)^{N^2}} \prod_{n=1}^{N-1} (n!)^2. \tag{2.19}$$

This last expression shows that if matrix A exists (that is if all its entries are finite) then it is invertible, since its determinant never vanishes (for finite ϕ). Factorization (2.4) therefore leads to an equivalent representation for the partition function, valid for all (nonvanishing) values of $\phi_\pm$.

Let us note that our choice of the constant $(-\mathrm{i})$ in the definition of matrix (2.3) can be easily explained now. Indeed, by considering a combination of (2.19) and its formal complex conjugate $(\mathrm{i} \to -\mathrm{i})$ one finds

$$\det \left[\frac{\partial^{j+k}}{\partial\phi^{j+k}} (\cot\phi + \alpha) \right]_{j,k=0}^{N-1} = \frac{\cos(N\phi) + \alpha \sin(N\phi)}{(\sin\phi)^{N^2}} \prod_{n=1}^{N-1} (n!)^2. \tag{2.20}$$

The expression $\cos(N\phi) + \alpha\sin(N\phi)$ possesses zeros in the complex plane of the variable ϕ unless $\alpha = \pm\mathrm{i}$. Thus, by choosing $\alpha = -\mathrm{i}$ (or, equivalently, $\alpha = \mathrm{i}$) the invertibility of matrix A is ensured.

With (2.19) taken into account the partition function now reads

$$Z_N = [\sin\phi_+]^{N^2}\, \mathrm{e}^{-\mathrm{i}N\phi_-} \det_N (I - A_-^{-1} A_+). \tag{2.21}$$

It is worth emphasizing that the partition function is, in fact, described only by the last factor, the first two factors having a trivial meaning, and disappearing with a mere redefinition of the vertex weights. Indeed, the first factor in (2.21) can be seen as the result of a common prefactor in all weights, and we get rid of it by changing the overall normalization of the weights. The second one is a 'boundary' factor specific to the DWBC choice for the six-vertex model; it can be removed by introducing a suitable asymmetry in weights w_5 and w_6, since any configuration contributing to the partition function (that is, satisfying both the 'ice rule' and the DWBC), is such that the numbers of vertices of these two types satisfy the condition $\#\mathsf{w}_6 - \#\mathsf{w}_5 = N$, Hence, by choosing the weights to be

$$\begin{aligned} &\mathsf{w}_1 = \mathsf{w}_2 = 1 &\qquad &\mathsf{w}_3 = \mathsf{w}_4 = \mathsf{b}/\mathsf{a} \\ &\mathsf{w}_5 = \mathrm{e}^{-\mathrm{i}(\lambda-\eta)}\mathsf{c}/\mathsf{a} &\qquad &\mathsf{w}_6 = \mathrm{e}^{\mathrm{i}(\lambda-\eta)}\mathsf{c}/\mathsf{a} \end{aligned} \tag{2.22}$$

with a, b and c given by (1.3), the partition function reduces to the sole determinant in (2.4).

The choice of the weights (2.22) has a simple interpretation in terms of the six-vertex model R-matrix, the matrix of local vertex states. In the standard notation the choice of the weights in the form (2.22) corresponds to the following R-matrix:

$$R(\nu) = \begin{pmatrix} 1 & & & \\ & \beta(\nu) & \mathrm{e}^{\mathrm{i}\nu}\gamma(\nu) & \\ & \mathrm{e}^{-\mathrm{i}\nu}\gamma(\nu) & \beta(\nu) & \\ & & & 1 \end{pmatrix} \tag{2.23}$$

where

$$\beta(\nu) = \frac{\sin\nu}{\sin(\nu+2\eta)} \qquad \gamma(\nu) = \frac{\sin 2\eta}{\sin(\nu+2\eta)} \qquad \nu = \lambda - \eta. \tag{2.24}$$

In this expression one easily recognizes the $U_q(\mathfrak{sl}_2)$-invariant R-matrix [21] which is, moreover, normalized to satisfy the 'unitarity condition':

$$R(\nu)(PR(-\nu)P) = I \tag{2.25}$$

with the permutation operator $P := R(0)$. Thus, the factorization (2.4) of the initial Hänkel determinant leads naturally to the representation of the partition function just as the sole determinant in (2.21), provided that the vertex weights are chosen according to the underlying quantum group symmetry of the model. From now on we shall assume this choice, equation (2.22), for the vertex weights, denoting the corresponding partition function as $\tilde{Z}_N$, given as

$$\tilde{Z}_N = \det{}_N(I - A_-^{-1} A_+). \tag{2.26}$$

It is worth mentioning here that almost everything above can also be extended to the case of the determinant formula of [2] for the partition function of the inhomogeneous model. Without giving any detail, let us just emphasize that previous considerations become even more transparent, since in that (more general) case matrix A is a Cauchy matrix. However, the homogeneous model possesses the further interesting property that the partition function can be expressed as the Fredholm determinant of an integral operator of integrable type, in the sense of [18]. This will be the subject of the next section.

3. The partition function as a Fredholm determinant

We shall now focus our attention on the determinant in (2.26). First of all we need to build the entries of matrix $A_-^{-1} = A^{-1}|_{\phi=\phi_-}$. This task can be achieved straightforwardly by borrowing standard techniques from the theory of orthogonal polynomials [20, 23]. Let us recall that, indeed, once the set of orthogonal polynomials associated with a given Hänkel matrix A with entries (2.5) is known, the entries of the corresponding inverse matrix A^{-1} can be evaluated from the function

$$\mathcal{K}_N(x, y) = \sum_{n=0}^{N-1} p_n(x) p_n(y) \tag{3.1}$$

simply in terms of partial derivatives:

$$A_{jk}^{-1} = \frac{1}{j!} \frac{\partial^j}{\partial x^j} \frac{1}{k!} \frac{\partial^k}{\partial y^k} \mathcal{K}_N(x, y) \bigg|_{x=0, y=0}. \tag{3.2}$$

The proof is based on the fact that function $\mathcal{K}_N(x, y)$ is the kernel of an integral operator, with respect to the measure $\omega(x)\,\mathrm{d}x$. This operator, by construction, projects over the subspace of polynomials of order less than N, and therefore acts as the identity operator on monomials $1, x, \ldots, x^{N-1}$:

$$\int_{-\infty}^{\infty} \mathcal{K}_N(x, y) y^m \omega(x)\,\mathrm{d}x = x^m \qquad m = 0, 1, \ldots, N-1. \tag{3.3}$$

With (3.3) taken into account, equation (3.2) can be verified directly:

$$\begin{aligned}
\sum_{k=0}^{N-1} A_{jk}^{-1} A_{km} &= \sum_{k=0}^{N-1} \frac{1}{j!} \frac{\partial^j}{\partial x^j} \frac{1}{k!} \frac{\partial^k}{\partial y^k} \mathcal{K}_N(x, y) \int_{-\infty}^{\infty} z^{k+m} \omega(z)\,\mathrm{d}z \bigg|_{x=0, y=0} \\
&= \frac{1}{j!} \frac{\partial^j}{\partial x^j} \int_{-\infty}^{\infty} \mathcal{K}_N(x, z) z^m \omega(z)\,\mathrm{d}z \bigg|_{x=0} \\
&= \frac{1}{j!} \frac{\partial^j}{\partial x^j} x^m \bigg|_{x=0} \\
&= \delta_{jm}.
\end{aligned} \tag{3.4}$$

In the case of matrix A (2.3) the entries of the inverse matrix A^{-1} are given by (3.2) with the function $\mathcal{K}_N(x,y)$ built from formula (3.1) in terms of orthogonal polynomials $p_n(x)$, defined in equation (2.17).

Generalizing the simple computation above one can evaluate the entries of the product matrix $A_-^{-1}A_+$:

$$[A_-^{-1}A_+]_{jm} = \frac{1}{j!}\frac{\partial^j}{\partial x^j}\int_{-\infty}^{\infty}\mathcal{K}_N^-(x,z)z^m\omega^+(z)\,\mathrm{d}z\bigg|_{x=0}. \tag{3.5}$$

Here the + or − superscripts denote the dependence on the variables $\phi_\pm$ respectively. The determinant in (2.26) can now be transformed as follows:

$$\begin{aligned}
\ln\det{}_N(I - A_-^{-1}A_+)&\\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_N(A_-^{-1}A_+)^n\\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathcal{K}_N^-(x_1,x_2)\mathcal{K}_N^-(x_2,x_3)\cdots\mathcal{K}_N^-(x_n,x_1)\prod_{l=1}^{n}\omega^+(x_l)\,\mathrm{d}x_l\\
&=: -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(\mathcal{V}_N)^n\\
&= \ln\det(\mathcal{I} - \mathcal{V}_N)
\end{aligned} \tag{3.6}$$

where $\mathcal{V}_N$ is the integral operator on the real axis with kernel

$$\mathcal{V}_N(x,y) = \mathcal{K}_N^-(x,y)\omega^+(y). \tag{3.7}$$

Using the Christoffel–Darboux identity, this kernel can be written as

$$\mathcal{V}_N(x,y) = \frac{\kappa_{N-1}^-}{\kappa_N^-}\frac{p_N^-(x)p_{N-1}^-(y) - p_{N-1}^-(x)p_N^-(y)}{x-y}\omega^+(y) \tag{3.8}$$

rendering the integrability (in the sense of [18]) of integral operator $\mathcal{V}_N$ manifest. Integral kernels of the form (3.8) are also known under the name of correlation kernels since they arise in expressions for eigenvalue correlation functions in the theory of random matrices. For discussion of the role and importance of this special class of integral operators in connection with correlation functions of integrable models and with the theory of random matrix, see [19, 25].

Taking into account formulae (2.10), (2.16), (2.17) and (2.18) we therefore get for the partition function of the model, with the choice (2.22) for its vertex weights, the following Fredholm determinant representation:

$$\tilde{Z}_N = \det(\mathcal{I} - \mathcal{V}_N) \tag{3.9}$$

where the kernel may be most explicitly written as

$$\begin{aligned}
\mathcal{V}_N(x,y) = \bigg\{&{}_2F_1\left(\left.\begin{matrix}-N,\mathrm{i}x/2\\1\end{matrix}\right|1-\mathrm{e}^{-2\mathrm{i}\phi_-}\right){}_2F_1\left(\left.\begin{matrix}-N+1,\mathrm{i}y/2\\1\end{matrix}\right|1-\mathrm{e}^{-2\mathrm{i}\phi_-}\right)\\
&-{}_2F_1\left(\left.\begin{matrix}-N+1,\mathrm{i}x/2\\1\end{matrix}\right|1-\mathrm{e}^{-2\mathrm{i}\phi_-}\right){}_2F_1\left(\left.\begin{matrix}-N,\mathrm{i}y/2\\1\end{matrix}\right|1-\mathrm{e}^{-2\mathrm{i}\phi_-}\right)\bigg\}\\
&\times\frac{N\mathrm{e}^{2\mathrm{i}N\phi_-}}{x-y}\frac{\mathrm{e}^{\phi_+ y}}{1-\mathrm{e}^{\pi y}+\mathrm{i}\,0}.
\end{aligned} \tag{3.10}$$

This representation is valid for $0 < \mathrm{Re}\,\phi_+ < \pi$ and arbitrary complex ϕ_-. Let us moreover underline that, in contrast to the original Hänkel determinant formula, the parameter N can be extended here from the set of positive integers to the whole complex plane.

We shall now discuss alternative forms for the representation we have just obtained. First, it should be noted that by shifting the integration contour and relabelling the integration variables in each integral in (3.6) this result can also be put in the form

$$\tilde{Z}_N = \det(\mathcal{I} - \zeta \mathcal{W}_N) \qquad \zeta = \mathrm{e}^{\mathrm{i}(\phi_- - \phi_+)} \tag{3.11}$$

with (to shorten the formulae we shall use a more compact notation in terms of polynomials)

$$\mathcal{W}_N(x, y) = N \frac{P_N^{(1/2)}(x; \phi_-)\, P_{N-1}^{(1/2)}(y; \phi_-) - P_{N-1}^{(1/2)}(x; \phi_-)\, P_N^{(1/2)}(y; \phi_-)}{x - y} \frac{\mathrm{e}^{2\phi_+ y}}{1 + \mathrm{e}^{2\pi y}}. \tag{3.12}$$

We see here that as a matter of fact the kernel is a real-valued function for real $\phi_\pm$ (since Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x; \phi)$ are real-valued for real x and ϕ).

The parametrization of the weights in the form (1.3) with real λ and η (hence, real $\phi_\pm$) is typical for the so-called disordered phase of the model [8]. Thus, the just obtained representation for the partition function, even if valid for arbitrary choice of vertex weights, can be regarded as 'adapted' to the disordered phase. The other two physical regimes are ferroelectric and antiferroelectric and they can be obtained by choosing $\phi_\pm$ to be purely imaginary.

To obtain corresponding representations it is sufficient to note that in the case of complex ϕ_+ the integrals in (3.6) can be evaluated by closing the integration contours upwards (downwards) in the complex plane of the variables $x_1, \ldots, x_n$ if $\operatorname{Im} \phi_+ > 0$ (if $\operatorname{Im} \phi_+ < 0$). As a result, each integral is given by a sum of residues at simple poles of the function $\omega(x)$ lying in the upper (respectively lower) complex half-plane. Considering the case of $\operatorname{Im} \phi_+ > 0$ (the case of $\operatorname{Im} \phi < 0$ leading to an essentially equivalent result) we obtain the following representation for the partition function:

$$\tilde{Z}_N = \det(\mathcal{I} - \tilde{\mathcal{V}}_N) \tag{3.13}$$

where $\tilde{\mathcal{V}}_N$ is the integral operator with discrete kernel

$$\begin{aligned} \tilde{\mathcal{V}}_N(x, y) = -\{ & M_N(x; 1, \mathrm{e}^{-2\tilde{\phi}_-}) M_{N-1}(y; 1, \mathrm{e}^{-2\tilde{\phi}_-}) \\ & - M_{N-1}(x; 1, \mathrm{e}^{-2\tilde{\phi}_-}) M_N(y; 1, \mathrm{e}^{-2\tilde{\phi}_-})\} \frac{N \mathrm{e}^{-2N\tilde{\phi}_-}}{x - y} \mathrm{e}^{-2\tilde{\phi}_+ y} \end{aligned} \tag{3.14}$$

whose 'integration' variables x, y take non-negative integer values; for $x = y$ the kernel is to be understood in the sense of the Christoffel–Darboux identity; in the last formula, the standard notation for Meixner polynomials

$$M_n(x; \beta, c) = {}_2F_1\left(\left. \begin{matrix} -n, -x \\ \beta \end{matrix} \right| 1 - \frac{1}{c} \right) \tag{3.15}$$

has been used [24]. The 'tilded' variables are defined as $\phi_\pm = \mathrm{i}\tilde{\phi}_\pm$. The representation (3.13) is valid for $\operatorname{Re} \tilde{\phi}_+ > 0$ and arbitrary complex $\tilde{\phi}_-$. The real values of $\tilde{\phi}_\pm$ correspond to ferroelectric ($\tilde{\phi}_\pm > 0$) and antiferroelectric ($\tilde{\phi}_+ > 0, \tilde{\phi}_- < 0$) phases of the model.

It is to be mentioned that representation (3.11) can also be obtained directly by employing the formula

$$\coth \tilde{\phi} = 2 \sum_{x=0}^{\infty} \mathrm{e}^{-2\tilde{\phi} x} \qquad \operatorname{Re} \tilde{\phi} > 0 \tag{3.16}$$

instead of (2.9) and repeating all considerations of the previous section. The appearance of Meixner polynomials is then quite obvious since for $|c| < 1$ they are subject to the orthogonality condition

$$\sum_{x=0}^{\infty} M_j(x; \beta, c) M_k(x; \beta, c) \frac{(\beta)_x}{x!} c^x = \frac{c^{-j}}{1 - c} \delta_{jk} \tag{3.17}$$

and upon setting $\beta = 1$ and $c = e^{-2\tilde{\phi}}$ the identification of the proper set of orthogonal polynomials is achieved.

We conclude this section by considering the so-called rational parametrization of the vertex weights, which corresponds to the case in which a, b and c are restricted by the condition $\mathsf{a} \pm \mathsf{b} = \mathsf{c}$. This regime can be obtained through a suitable limit from vertex weights (1.3). Namely, depending on the choice of the sign in this restriction, one should just substitute $\lambda, \eta \to \epsilon\lambda, \epsilon\eta$ (for plus sign) or $\lambda, \eta \to \pi/2 - \epsilon\lambda, \pi/2 - \epsilon\eta$ (for minus sign) in (1.3) and take the limit $\epsilon \to 0$, after renormalization of the weights by a factor $1/\epsilon$. We should of course recover in this limit the corresponding result of [20].

The Hänkel determinant formula for the partition function of the model with rational weights is simply given by formula (1.4) with the rational functions λ, η instead of sine functions: $\sin(*) \to (*)$. The Fredholm determinant representation for the partition function in this case involves Laguerre polynomials; the 'rational limit' of the kernel (3.12) can be easily found using

$$\lim_{\epsilon\to 0} P_n^{(1/2)}(x/\epsilon; \epsilon\phi) = L_n(-2\phi x) \qquad \lim_{\epsilon\to 0} \frac{e^{\epsilon\phi_+(y/\epsilon)}}{1 + e^{\pi(y/\epsilon)}} = e^{\phi_+ y}\theta(-y) \tag{3.18}$$

where $L_n(x)$ is the Laguerre polynomial, and $\theta(x)$ is the Heaviside step-function. The same result can be obtained from (3.14), too, with the discrete measure turning into the continuous one in the standard way, when performing the rational limit. Explicitly, the partition function in the rational case is given by the Fredholm determinant of the integral operator on the real positive half-axis, defined by the kernel

$$\mathcal{V}_N(x, y) = -N \frac{L_N(\xi x)L_{N-1}(\xi y) - L_{N-1}(\xi x)L_N(\xi y)}{x - y} e^{-y} \tag{3.19}$$

where $\xi = \phi_-/\phi_+$. The result of [20] is thus reproduced in the rational limit, which corresponds there to the case $q = 1$.

4. The finite-size determinant representation

The determinant in (2.26) can also be written as that of some $N \times N$ symmetric matrix whose entries are simply connected with the kernel of the integral operator in the Fredholm determinant representation. One can insert (3.1) in the second line of (3.6) and obtain

$$\begin{aligned}
&\ln \det_N(I - A_-^{-1}A_+) \\
&\quad = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathcal{K}_N^-(x_1, x_2)\mathcal{K}_N^-(x_2, x_3) \cdots \mathcal{K}_N^-(x_n, x_1) \prod_{l=1}^{n} \omega^+(x_l)\, dx_l \\
&\quad = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k_1,\ldots,k_n=0}^{N-1} V_{k_1k_2}V_{k_2k_3}\ldots V_{k_nk_1} \\
&\quad = \ln \det_N(I - V)
\end{aligned} \tag{4.1}$$

where entries of the matrix V are

$$V_{jk} = \int_{-\infty}^{\infty} p_j^-(x)p_k^-(x)\omega^+(x)\, dx. \tag{4.2}$$

Clearly, this identity is a consequence of the fact that the matrices $A_-^{-1}A_+$ and V are related by some similarity transformation.

Hence, the partition function admits also the representation

$$\tilde{Z}_N = \det_N(I - V). \tag{4.3}$$

It turns out that in contrast to entries of matrix H, entering the original Hänkel determinant representation (1.4), those of the matrix V can be computed explicitly. It is convenient (in analogy with (3.11)) to introduce the matrix W by

$$V = \zeta W \qquad \zeta = \mathrm{e}^{\mathrm{i}(\phi_- - \phi_+)} = \mathrm{e}^{-2\mathrm{i}\eta}. \tag{4.4}$$

Entries of W are defined by the formula

$$W_{jk} = 2\sin\phi_- \int_{-\infty}^{\infty} P_j^{(1/2)}(x;\phi_-)\, P_k^{(1/2)}(x;\phi_-)\, \frac{\mathrm{e}^{2x\phi_+}}{1+\mathrm{e}^{2\pi x}}\,\mathrm{d}x. \tag{4.5}$$

As shown in detail in the appendix, the integral can be evaluated in closed form. Introducing, for the sake of convenience, the notation

$$\beta = \frac{\sin\phi_-}{\sin\phi_+} = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)} \qquad \gamma = \frac{\sin(\phi_+ - \phi_-)}{\sin\phi_+} = \frac{\sin 2\eta}{\sin(\lambda+\eta)} \tag{4.6}$$

(note that β and γ are precisely the quantities entering the $U_q(\mathfrak{sl}_2)$-invariant R-matrix, see equations (2.23) and (2.24), $\beta = \beta(\nu)$, $\gamma = \gamma(\nu)$) the result reads

$$\begin{aligned} W_{jk} &= \gamma^{j+k+1} \sum_{n=0}^{\min(j,k)} \binom{j}{n}\binom{k}{n}\left(\frac{\beta}{\gamma}\right)^{2n+1} \\ &= \beta\gamma^{j+k}\,{}_2F_1\left(\begin{matrix} -j,-k \\ 1\end{matrix}\,\middle|\,\frac{\beta^2}{\gamma^2}\right). \end{aligned} \tag{4.7}$$

It is worth recalling that the representation (4.3) is for the partition function with the weights (2.22); the original normalization can be achieved by multiplying the RHS of (4.3) by the factor $[\sin(\lambda+\eta)]^{N^2}\,\mathrm{e}^{\mathrm{i}(\lambda-\eta)N}$.

The second expression in (4.7) shows that the entries of matrix W are, in fact, Meixner polynomials, see (3.15). On the other hand, from the first expression in (4.7) it follows that the matrix W can be written as a product of much simpler matrices. Indeed, introducing $N \times N$ matrices

$$(J_+)_{nm} = n\delta_{n-1,m} \qquad (J_0)_{nm} = (n+1/2)\delta_{n,m} \qquad (J_-)_{nm} = (n+1)\delta_{n+1,m} \tag{4.8}$$

and taking into account that

$$(\exp\{\gamma J_+\})_{nm} = \gamma^{n-m}\binom{n}{m} \qquad J_- = (J_+)^T \tag{4.9}$$

one obtains

$$W = \exp\{\gamma J_+\}\beta^{2J_0}\exp\{\gamma J_-\}. \tag{4.10}$$

Thus, the matrix W is defined by its Gauss decomposition, i.e., as a product of a lower-triangular, a diagonal and an upper-triangular matrix, respectively.

The matrix W simplifies considerably in the limit $N \to \infty$. This is a consequence of the fact that the semi-infinite dimensional matrices $J_{\pm,0}$ with entries (4.8) satisfy $\mathfrak{su}(1,1)$ algebra commutation relations

$$[J_-, J_+] = 2J_0 \qquad [J_\pm, J_0] = \mp J_\pm. \tag{4.11}$$

Standard techniques (widely used for instance in the theory of generalized coherent states [26]) can now be applied to (4.10), with the following result:

$$W = \exp\{2\eta K\} \qquad (N = \infty) \tag{4.12}$$

where the semi-infinite dimensional matrix K is

$$K = \frac{1}{\sin\nu}(J_- + J_+ - 2\cos\nu J_0). \tag{4.13}$$

Here (and below) we use 'mixed' set of variables, $\nu = \phi_- = \lambda - \eta$ and η (indeed the most natural ones for the $U_q(\mathfrak{sl}_2)$-invariant R-matrix, see equations (2.23)–(2.25)).

The matrix K is the Jacobi matrix corresponding to the recurrence relation of Meixner–Pollaczek, Laguerre or Meixner polynomials depending on whether $|\cos\nu| < 1$, $|\cos\nu| = 1$ or $|\cos\nu| > 1$, respectively. These polynomials are therefore eigenfunctions of K with eigenvalue x, and the spectrum of K is thus given by the support of the measure appearing in the orthogonality condition of the corresponding polynomials, which is $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{Z}_+$, respectively (i.e., it is continuous in the first and second cases and discrete in the third one). From this point of view the presence of these same polynomials in the kernel of the integral operator in the Fredholm determinant representations is obvious: the integral operator is just the semi-infinite dimensional matrix $\Pi_N \exp\{2\eta K\}$ reexpressed in the basis of the eigenfunctions of K; here the N-dimensional projector Π_N is the matrix with its first N diagonal entries equal to one, and all other entries equal to zero.

Another point which is to be discussed is that formula (4.12) tells us that in the case where $N = \infty$, matrix W (and hence V) is just the exponential of some 'simple' matrix. The analogy which comes into mind after looking at formulae (4.3) and (4.12) is with a typical result of calculation of traces of certain class of operators acting over the Fock space of N fermions. Namely, given a matrix A, and the corresponding quantum operator $\hat{\boldsymbol{A}}$, bilinear in canonical fermion operators, built from matrix A as follows

$$\hat{\boldsymbol{A}} = \sum_{n,m=0}^{N-1} \hat{c}_{n+1}^{\dagger} A_{nm} \hat{c}_{m+1} \qquad \hat{c}_n^{\dagger}\hat{c}_m + \hat{c}_n\hat{c}_m^{\dagger} = \delta_{nm} \tag{4.14}$$

it is well known [27] that

$$\mathrm{Tr}[\exp\hat{\boldsymbol{A}}] = \det_N(I + \exp A). \tag{4.15}$$

To make a connection with formula (4.3), it is to be mentioned that the minus sign in RHS in (4.15) can be acquired by considering supertraces instead of traces; the supertrace is defined as $\mathrm{Str}[*] := \mathrm{Tr}[(-1)^{\hat{\boldsymbol{N}}} *]$, where $\hat{\boldsymbol{N}}$ denotes the fermion number operator: $\hat{\boldsymbol{N}} = \sum_{n=1}^{N} \hat{c}_n^{\dagger}\hat{c}_n$. However, since the fermion number operator commutes with any operator of the form (4.14), one can consider just traces when dealing with the exponentials of such operators. Taking this into account we can therefore write the following trace formula for the partition function:

$$\tilde{Z}_N = \mathrm{Tr}[\exp\{2\eta\hat{\boldsymbol{L}}_N\}] \tag{4.16}$$

with

$$\hat{\boldsymbol{L}}_N = \hat{\boldsymbol{K}}_N + \mathrm{i}\frac{\pi - 2\eta}{2\eta}\hat{\boldsymbol{N}} \tag{4.17}$$

where the operator $\hat{\boldsymbol{K}}_N$ is built from the matrix

$$K_N = \frac{1}{2\eta}\ln W. \tag{4.18}$$

Note that in contrast to the entries of matrix W those of matrix K_N (and hence of L_N) essentially depend on N as indicated by the notation. Operator $\hat{\boldsymbol{K}}_N$, though bilinear in fermions, is essentially nonlocal, thus making the entries of matrix K_N for finite N rather complicated quantities. However, formula (4.12) implies that the matrix K_N simplifies considerably in the case $N = \infty$, and the corresponding operator $\hat{\boldsymbol{K}} := \hat{\boldsymbol{K}}_\infty$ becomes a local one. Explicitly we obtain

$$\hat{\boldsymbol{K}} = \sum_{n=1}^{\infty} n\hat{\boldsymbol{H}}_{n,n+1} \tag{4.19}$$

where

$$\hat{\boldsymbol{H}}_{n,n+1} = \frac{1}{\sin\nu}\left\{\hat{c}_n^\dagger \hat{c}_{n+1} + \hat{c}_{n+1}^\dagger \hat{c}_n - \cos\nu\left(\hat{c}_n^\dagger \hat{c}_n + \hat{c}_{n+1}^\dagger \hat{c}_{n+1}\right)\right\}. \tag{4.20}$$

We recognize in this expression for operator $\hat{\boldsymbol{K}}$ what is known in the literature as a boost, or ladder, operator. To be precise $\hat{\boldsymbol{K}}$ is the positive half-axis part $\hat{\boldsymbol{K}} = \hat{\boldsymbol{B}}^{(+)}$ of the total boost operator $\hat{\boldsymbol{B}} = \hat{\boldsymbol{B}}^{(-)} + \hat{\boldsymbol{B}}^{(+)}$, with $[\hat{\boldsymbol{B}}^{(+)}, \hat{\boldsymbol{B}}^{(-)}] = 0$, see [28]. Operator $\hat{\boldsymbol{B}}$ is the boost operator for the model described by the following Hamiltonian:

$$\hat{\boldsymbol{H}} = \sum_n \hat{\boldsymbol{H}}_{n,n+1} = \frac{1}{\sin\nu}\hat{\boldsymbol{H}}_0 - 2\cot\nu\,\hat{\boldsymbol{N}} \tag{4.21}$$

where $\hat{\boldsymbol{H}}_0$ is the hopping term. From this formula it is clear that ν, the spectral parameter in the six-vertex model, is here playing the role of a chemical potential rather than a coupling constant. Nevertheless, all the construction above has led us to a description which is quite analogous to what we have in the corner transfer matrix formalism [8], or to a more general extent, in the vertex operator approach to integrable models [22] and angular quantization method in integrable quantum field theory (see, e.g., [29] and references therein).

As previously explained, in the case of finite N operator $\hat{\boldsymbol{K}}_N$ is nonlocal (though free-fermionic). One can however still write the partition function in terms of the product of exponentials of local operators. As a generalization of (4.15) the following is also valid

$$\operatorname{Tr}\left[\prod_i \exp\hat{\boldsymbol{A}}_i\right] = \det{}_N\left(I + \prod_i \exp A_i\right) \tag{4.22}$$

where operators $\hat{\boldsymbol{A}}_i$ are constructed out of matrices A_i through formula (4.14). We can therefore write the following 'reconstruction' trace formula for the partition function

$$\tilde{Z}_N = \operatorname{Tr}[(-\mathrm{e}^{-2\mathrm{i}\eta})^{\hat{\boldsymbol{N}}} \exp\{\gamma\hat{\boldsymbol{J}}_+\}\beta^{2\hat{\boldsymbol{J}}_0}\exp\{\gamma\hat{\boldsymbol{J}}_-\}] \tag{4.23}$$

with

$$\hat{\boldsymbol{J}}_+ = \sum_{n=1}^{N-1} n c_{n+1}^\dagger c_n \qquad \hat{\boldsymbol{J}}_0 = \sum_{n=1}^{N}(n-1/2)c_n^\dagger c_n \qquad \hat{\boldsymbol{J}}_- = (\hat{\boldsymbol{J}}_+)^\dagger. \tag{4.24}$$

The reconstruction formula (4.23) expresses the partition function of the six-vertex model as the trace of a quantum operator built out of fermions, or, modulo Jordan–Wigner transformation, of spin-$1/2$ operators. Note that, in fact, we have derived these expressions for the partition function starting from the Hänkel determinant representation, previously obtained within the quantum inverse scattering method. The question which arises from the considerations presented here, is how representation (4.23) in terms of fermions (or spins), which are to be interpreted as effective degrees of freedom, could be extracted directly from the basic definition of the model in terms of vertex configurations. An answer to this question might suggest alternative approaches to the open problem of calculation of correlation functions for the model.

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Appendix

The integral defining the entries of matrix W in (4.5) is a particular case of the integral

$$I_{nm}^{(\lambda)}(\tau,\omega;\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_n^{(\lambda)}(x;\tau)\, P_m^{(\lambda)}(x;\omega)\, |\Gamma(\lambda+\mathrm{i}x)|^2\, \mathrm{e}^{(2\phi-\pi)x}\, \mathrm{d}x \tag{A.1}$$

where $\phi \in (0,\pi)$ and λ is assumed to be real and positive, $\lambda > 0$. These restrictions are important for convergence of the integral. Our aim here is to prove that for arbitrary τ and ω the quantity $I_{nm}^{(\lambda)}(\tau,\omega;\phi)$ has the following expression:

$$\begin{aligned} I_{nm}^{(\lambda)}(\tau,\omega;\phi) &= \frac{\Gamma(2\lambda+n)\Gamma(2\lambda+m)}{(2\sin\phi)^{2\lambda}\Gamma(2\lambda)n!m!}\left[\frac{\sin(\tau-\phi)}{\sin\phi}\right]^n\left[\frac{\sin(\omega-\phi)}{\sin\phi}\right]^m \\ &\quad\times {}_2F_1\left(\begin{matrix}-n,-m\\ 2\lambda\end{matrix}\,\middle|\,\frac{\sin\tau\sin\omega}{\sin(\tau-\phi)\sin(\omega-\phi)}\right). \end{aligned} \tag{A.2}$$

Equality (A.2) is a consequence of the orthogonality condition for the Meixner–Pollaczek polynomials and of the identity

$$P_n^{(\lambda)}(x;\tau) = \sum_{k=0}^{n} \frac{\Gamma(n+2\lambda)}{\Gamma(k+2\lambda)(n-k)!}\frac{[\sin(\tau-\phi)]^{n-k}(\sin\tau)^k}{(\sin\phi)^n} P_k^{(\lambda)}(x;\phi) \tag{A.3}$$

which can be viewed as the extension of an analogous formula for Laguerre polynomials (see, e.g., [30], section 10.12, equation (40)). Indeed, using identity (A.3) for both polynomials under the integration sign and employing the orthogonality condition (2.14) one immediately obtains

$$\begin{aligned} I_{nm}^{(\lambda)}(\tau,\omega;\phi) &= \frac{[\sin(\tau-\phi)]^n[\sin(\omega-\phi)]^m}{2^{2\lambda}(\sin\phi)^{n+m+2\lambda}} \\ &\quad\times \sum_{k=0}^{\min(n,m)} \frac{\Gamma(n+2\lambda)\Gamma(m+2\lambda)}{\Gamma(k+2\lambda)(n-k)!(m-k)!k!}\left[\frac{\sin\tau\sin\omega}{\sin(\tau-\phi)\sin(\omega-\phi)}\right]^k. \end{aligned} \tag{A.4}$$

Rewriting the finite sum here as a truncated hypergeometric series results in expression (A.2).

Let us prove now the identity (A.3). To simplify as much as possible the combinatorics, let us consider the three-term relation satisfied by the Meixner–Pollaczek polynomials:

$$(n+1)P_{n+1}^{(\lambda)}(x;\phi) - [2x\sin\phi + 2(n+\lambda)\cos\phi]P_n^{(\lambda)}(x;\phi) + (n+2\lambda-1)P_{n-1}^{(\lambda)}(x;\phi) = 0. \tag{A.5}$$

Let us define

$$S_n(x;\phi) = (\sin\phi)^{-n} P_n^{(\lambda)}(x;\phi)\,. \tag{A.6}$$

Recurrence relation (A.5) takes the form

$$\begin{aligned} &(n+1)S_{n+1}(x;\phi) - 2(n+\lambda)\cot\phi\, S_n(x;\phi) \\ &\qquad + (n+2\lambda-1)[1+(\cot\phi)^2]S_{n-1}(x;\phi) = 2xS_n(x;\phi). \end{aligned} \tag{A.7}$$

From this relation it is clear that $S_n(x;\phi)$ depends on ϕ only through $\cot\phi$, and, moreover, it is a polynomial of order n in $\cot\phi$. It is useful to consider the case $\phi = \pi/2$ so that $\cot\phi = 0$. Denoting $S_n(x) = S_n(x;\pi/2)$, in this case one has

$$(n+1)S_{n+1}(x) + (n+2\lambda-1)S_{n-1}(x) = 2xS_n(x). \tag{A.8}$$

Let us consider the semi-infinite dimensional matrices

$$(J_-)_{nm} = (n+1)\delta_{n+1,m} \qquad (J_0)_{nm} = (n+\lambda)\delta_{n,m} \qquad (J_+)_{nm} = (n+2\lambda-1)\delta_{n-1,m} \tag{A.9}$$

where, as in the main text of the paper, $n, m \in \{0, 1, 2, \ldots\}$. These matrices obey $\mathfrak{su}(1,1)$ algebra commutation relations (4.11). Modulo a diagonal similarity transformation, they correspond to the standard matrix realization of the positive discrete representation $\mathcal{D}^{(+)}(\lambda)$ of $\mathfrak{su}(1,1)$. In terms of matrices (A.9) relations (A.7) and (A.8) read

$$\begin{aligned} &[J_- - 2\cot\phi\, J_0 + (1+(\cot\phi)^2)J_+]\vec{S}(x;\phi) = 2x\vec{S}(x;\phi) \\ &[J_- + J_+]\vec{S}(x) = 2x\vec{S}(x) \end{aligned} \tag{A.10}$$

where the nth component of $\vec{S}(x,\phi)$ is just $S_n(x,\phi)$, and $\vec{S}(x) \equiv \vec{S}(x,\pi/2)$. Using the commutation relations of $\mathfrak{su}(1,1)$ algebra it can be straightforwardly checked that the following relation is valid

$$\exp\{\alpha J_+\}[J_- + J_+] = [J_- - 2\alpha J_0 + (1+\alpha^2)J_+]\exp\{\alpha J_+\}. \tag{A.11}$$

Thus, by identifying $\alpha = \cot\phi$ we obtain

$$\vec{S}(x;\phi) = \exp\{\cot\phi\, J_+\}\vec{S}(x). \tag{A.12}$$

This is a key identity to prove relation (A.3). Indeed relation (A.12) implies that

$$\vec{S}(x;\tau) = \exp\{(\cot\tau - \cot\phi)J_+\}\vec{S}(x;\phi). \tag{A.13}$$

Then, taking into account that

$$(\exp\{\alpha J_+\})_{nm} = \frac{\Gamma(n+2\lambda)}{\Gamma(m+2\lambda)(n-m)!}\alpha^{n-m} \tag{A.14}$$

one obtains

$$S_n(x;\tau) = \sum_{m=0}^{n} \frac{\Gamma(n+2\lambda)}{\Gamma(m+2\lambda)(n-m)!}(\cot\tau - \cot\phi)^{n-m} S_m(x;\phi). \tag{A.15}$$

Rewriting the last equation in terms of the standard Meixner–Pollaczek polynomials, see (A.6), one arrives finally at identity (A.3), which is thus proved.

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